4.1.1

线性性: $(X_1+X_2)(\{f\}+\{g\}) = X_1(\{f\}+\{g\}) + X_2(\{f\}+\{g\})$

$= X_1\{f\} + X_2\{f\} + X_1\{g\} + X_2\{g\}$

$= (X_1+X_2)\{f\} + (X_1+X_2)\{g\}$

$(X_1+X_2)(c\{f\}) = c(X_1\{f\}+X_2\{f\}) = c(X_1+X_2)\{f\}$

$(cX_1)(\{f\}+\{g\}) = c\cdot X_1(\{f\}+\{g\}) = cX_1\{f\} + cX_1\{g\} = (cX_1)\{f\} + (cX_1)\{g\}$

$(c_1X_1)(c_2\{f\}) = c_1c_2X_1\{f\} = c_2(c_1X_1)\{f\}$

导性: $(X_1+X_2)(\{f\}\cdot\{g\}) = X_1(\{f\}\cdot\{g\}) + X_2(\{f\}\cdot\{g\}) = g\cdot X_1f + f\cdot X_1 g + g\cdot X_2 f + f\cdot X_2 g$

$= g\cdot(X_1+X_2)f + f\cdot(X_1+X_2)g$

$(cX_1)(\{f\}\cdot\{g\}) = cX_1(\{f\}\cdot\{g\}) = c\cdot(g\cdot X_1 f + f\cdot X_1 g) = g\cdot(cX_1)f + f\cdot(cX_1)g$

$\Rightarrow$ $X_1+X_2,\ cX \in T_p(M)$ 且满足向量空间各个条件 $\Rightarrow$ 向量空间

4.1.3.

① $F_{*p}(X_p)(f+g) = X_p((f+g)\circ F) = X_p(f\circ F+g\circ F) = X_p(f\circ F) + X_p(g\circ F) = F_{*p}(X_p)f + F_{*p}(X_p)g$

② $F_{*p}(X_p)(cf) = X_p((cf)\circ F) = X_p(c(f\circ F)) = c\cdot X_p(f\circ F) = c\cdot F_{*p}(X_p)f$

③ $F_{*p}(X_p)(f\cdot g) = X_p((f\cdot g)\circ F) = X_p((f\circ F)(g\circ F)) = f\circ F(p)X_p(g\circ F) + g\circ F(p)X_p(f\circ F)$

$= f(F(p))F_{*p}(X_p)g + g(F(p))F_{*p}(X_p)f$

$\therefore\ F_{*p}(X_p) \subset T_{F(p)}(M)$

$F_{*p}(X_1+X_2)f = (X_1+X_2)(f\circ F) = X_1(f\circ F) + X_2(f\circ F) = F_{*p}(X_1)f + F_{*p}(X_2)f$

$F_{*p}(cX_1)f = (cX_1)(f\circ F) = c\cdot X_1(f\circ F) = cF_{*p}(X_1)f$

$\Rightarrow$ 线性映射

4.1.6

① $\forall X_p \in T_p(M_1)$ $\forall f \in F_{G\circ F(p)}$

$(G\circ F)_{*p}(X_p)f = X_p(f\circ G\circ F) = F_{*p}(X_p)(f\circ G) = G_{*F(p)}(F_{*p}(X_p))f = G_{*F(p)}\circ F_{*p}(X_p)f$

$\Rightarrow (G\circ F)_{*p} = G_{*F(p)}\circ F_{*p}$

② $F_{*p}\left(\frac{\partial}{\partial x^i}\right) = \sum_{i=1}^{n}\frac{\partial y^i}{\partial x^i}\frac{\partial}{\partial y^i}$

$G_{*F(p)}\left(\frac{\partial}{\partial y^i}\right) = \sum_{j=1}^{n}\frac{\partial z^j}{\partial y^i}\cdot\frac{\partial}{\partial z^j}$

$(G\circ F)_{*p}\left(\frac{\partial}{\partial x^i}\right) = \sum_{j=1}^{n}\frac{\partial z^j}{\partial x^i}\cdot\frac{\partial}{\partial z^j}$

$\Rightarrow (G\circ F)_{*p} = G_{*F(p)}\circ F_{*p}$

5.1.3. (1) $X^{**}(X_1^*+X_2^*) = (X_1^*+X_2^*)(X) = X_1^*(X) + X_2^*(X)$

$X^{**}(\lambda X^*) = (\lambda X^*)(X) = cX^*(X) = cX^{**}(X^*)$

$\Rightarrow$ 线性映射

(2) 设 $f_x \in X^{**}$  $\varphi: x \to f_x$

$f_{x_1+x_2}(\alpha) = \alpha(x_1)+\alpha(x_2) = (f_{x_1}+f_{x_2})(\alpha)$

$f_{\lambda x_1}(\alpha) = \alpha(\lambda x_1) = \lambda f_{x_1}(\alpha)$

$\Rightarrow \varphi(x_1+x_2) = \varphi(x_1)+\varphi(x_2)$ 且 $\varphi(\lambda x_1) = \lambda \varphi(x_1)$

$\frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_n}$ 为 $T_p(M)$ 的基    $dx_1 \cdots dx_n$ 为 $T_p^*M$ 的基

$\Rightarrow f_{\frac{\partial}{\partial x_i}}(dx_j) = \delta_{ij}$    $\Rightarrow f_{\frac{\partial}{\partial x_i}}$ 为 $T_p(M)^{**}$ 的一组基 为对偶基

$\Rightarrow \varphi$ 为同构